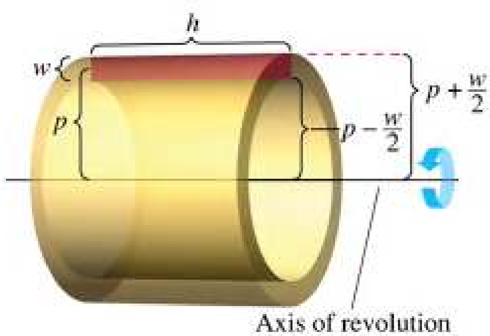
The Shell Method is an alternative method for finding the volume of a solid of revolution. This method is called the **shell method** because it uses cylindrical shells.

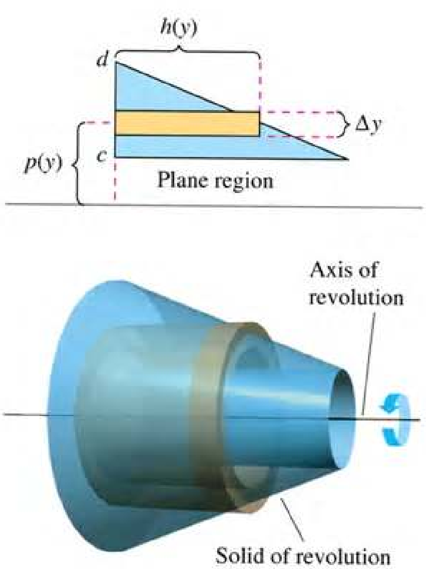


Consider a representative rectangle as shown in the figure, where *w* is the width of the rectangle, *h* is the height of the rectangle, and *p* is the distance between the axis of revolution and the *center* of the rectangle. When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness *w*. To find the volume of this shell, consider two cylinders. The radius of the larger cylinder corresponds to the outer radius of the shell, and the radius of the smaller cylinder corresponds to the inner radius of the shell. Because *p* is the average radius of the shell, you know the **outer radius is *p+(w/2)*** and the **inner radius is *p-(w/2)***.

So, the volume of the shell is

$$\begin{aligned}\text{Volume of shell} &= \text{(volume of cylinder)} - \text{(volume of hole)}\\ &= \pi\left(p+\frac{w}{2}\right)^2 h-\pi\left(p-\frac{w}{2}\right)^2 h\\ &= 2\pi phw\\ &= 2\pi\,\text{(average radius)(height)(thickness)}\end{aligned}$$

You can use this formula to find the volume of a solid of revolution.



Assume that the plane region in the figure to the left is revolved a bout a line to form the indicated solid. If you consider a horizontal rectangle of width Δy, then as the plane region is revolved about a line parallel to the x-axis, the rectangle generates a representative shell whose volume is

$\Delta V = 2\pi[p(y)h(y)]\Delta y$.

You can approximate the volume of the solid by *n* such shells of thickness Δy, height $h(y_i)$, and average radius $p(y_i)$.

The approximation appears to get better as $n \to \infty$. So the solid of the volume is

$$\text{Volume of solid} = 2\pi\int_c^d [p(y)h(y)]\partial y.$$

The Shell Method

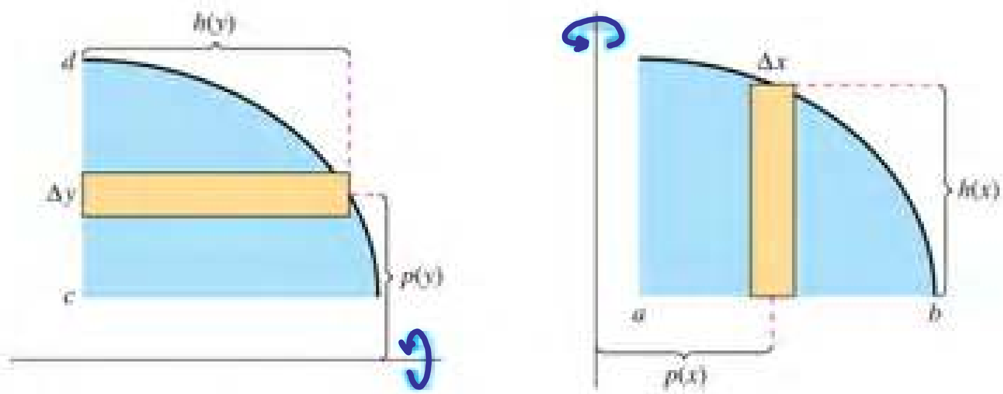
To find the volume of a solid of revolution with the **shell method,** use one of the following, as shown in Figure 7.29.

Horizontal Axis of Revolution

$$\text{Volume} = V = 2\pi\int_c^d p(y)h(y)\,dy$$

Vertical Axis of Revolution

$$\text{Volume} = V = 2\pi\int_a^b p(x)h(x)\,dx$$



Horizontal axis of revolution

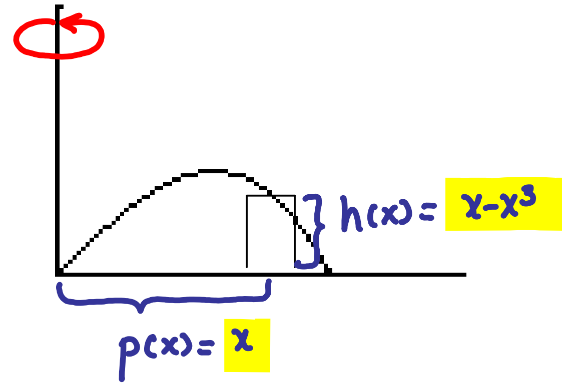
Vertical axis of revolution

SAMPLE PROBLEMS

1. Find the volume of the solid of revolution formed by revolving the region bounded by

$$y = x - x^3$$

and the x-axis $(0 \le x \le 1)$ about the y-axis.



$$V = 2\pi \int_0^1 x(x - x^3)\,dx = 2\pi \int_0^1 x^2 - x^4\,dx$$

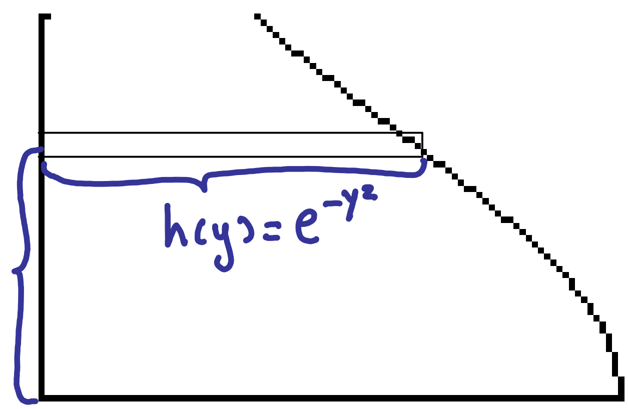
$$= 2\pi \left(\frac{1}{3}x^3 - \frac{1}{5}x^5 \Big]_0^1 \right) = 2\pi \left(\frac{1}{3} - \frac{1}{5} \right) =$$

$$= 2\pi \left(\frac{2}{15} \right) \qquad \boxed{V = \frac{4}{15}\pi}$$

2. Find the volume of the solid of revolution formed by revolving the region bounded by the graph of

$$x = e^{-y^2}$$

and the y-axis $(0 \le y \le 1)$ about the x-axis.

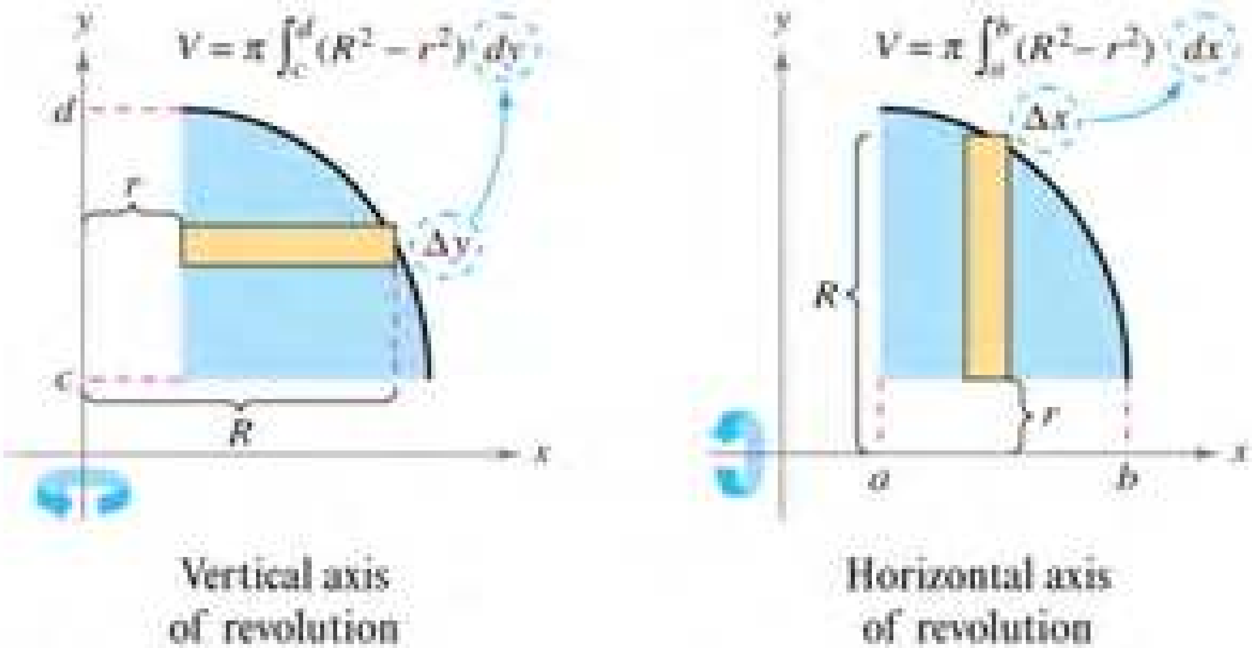


$p(y) = y$

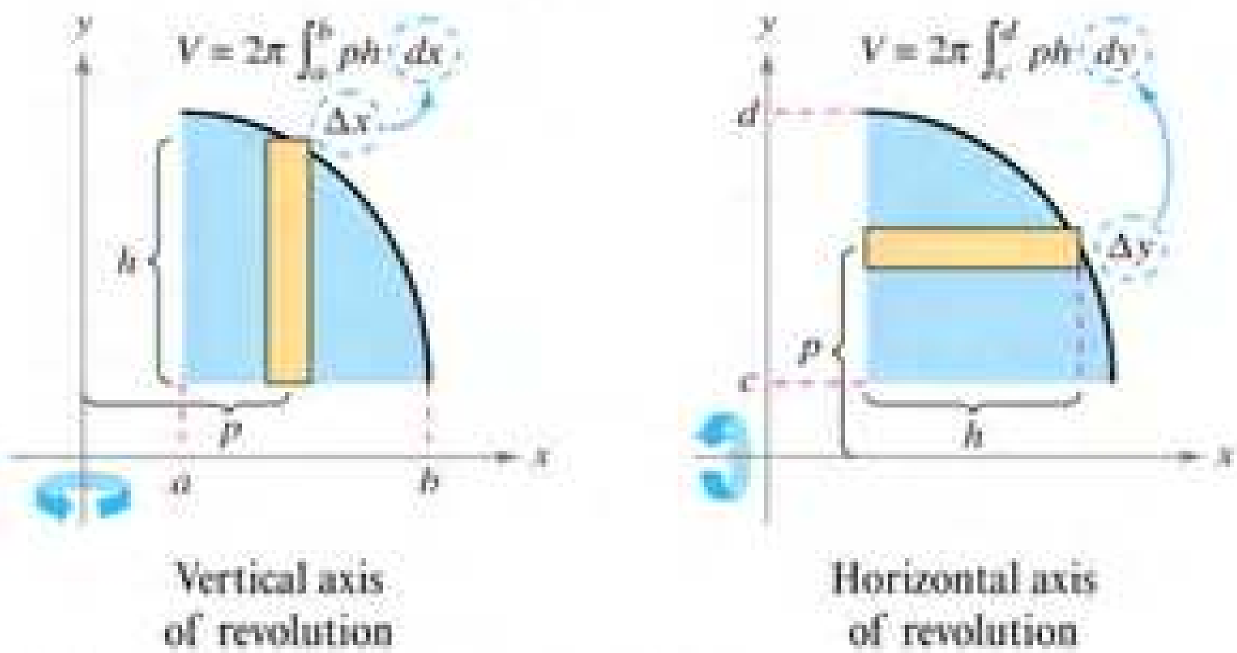
$$V = 2\pi \int_0^1 y e^{-y^2} dy = -\pi \int_0^1 -2y\, e^{-y^2} dy$$

$$= -\pi \left(e^{-y^2} \Big]_0^1 \right) = -\pi \left(e^{-1} - e^0 \right) = \pi \left(1 - 1/e \right) = \boxed{\frac{e-1}{e}\pi}$$

COMPARISON OF DISK AND SHELL METHODS



Disk method: Representative rectangle is perpendicular to the axis of revolution.



Shell method: Representative rectangle is parallel to the axis of revolution.